THE IRREDUCIBLE CASE OF THE CUBIC EQUATION.

By Alston Hamilton.

The most general form of the equation of the third degree is

$$x^3 + Ax^2 + Bx + C = 0.$$

In this, place $$x = y - \frac{A}{3}.$$

This gives the following equation involving y:

$$y^3 + py + q = 0,$$

where $$p = \frac{3B - A^2}{3},$$

and $$q = \frac{2A^3}{27} - \frac{AB}{3} + C.$$

Cardan's Rule applies to the equation $y^3 + py + q = 0$, and the result is satisfactorily obtainable when

$$\frac{q^2}{4} + \frac{p^3}{27} \geqq 0.$$

This inequality is satisfied whenever p is a positive quantity, and also when p is both negative and numerically less than or equal to $\frac{3}{2}\sqrt[3]{2q^2}$.

If Cardan's Rule "fails" we have the "Irreducible Case," so called because it involves roots of imaginary quantities the extraction of which is again dependent upon cubic equations.

For the sake of greater clearness we replace p, which in irreducible cases has been shown to be essentially negative, by $-p'$.

Then $$y^3 - p'y + q = 0. \qquad (1)$$

Now place $$y = z\sqrt{p'}.$$

Then, by substitution in equation (1)

$$z^3 - z + \frac{q}{p'\sqrt{p'}} = 0.$$

Place $$\frac{q}{p'\sqrt{p'}} = q';$$

then $$z^3 - z + q' = 0.$$

Enter the table on page 45 with the numerical value of q' as an argument and by interpolation find z. Take the sign of z the opposite of that of q'. Then

$$x = y - \frac{A}{3} = z\sqrt{p'} - \frac{A}{3}.$$

This is the outline of the *use* of the method. An example follows, and then follows the theory on which the preceding method is based.

EXAMPLE.

$$x^3 + 3x^2 - 5x + 1 = 0.$$

Here $$p = \frac{3B - A^2}{3} = -8,$$

and $$q = \frac{2A^3}{27} - \frac{AB}{3} + C = +8;$$

$\frac{q^2}{4} + \frac{p^3}{27} = -2\frac{26}{27}$, which < 0, and Cardan's Rule "fails."

Place $$y = x + \frac{A}{3} = x + 1,$$

then $$y^3 - 8y + 8 = 0.$$

Place $$y = z\sqrt{p'} = z\sqrt{8},$$

then $$z^3 - z + \frac{1}{4}\sqrt{2} = 0,$$

or $$z^3 - z + .353553 = 0.$$

Now refer to the table, and we find that

for $z = 1.144$ . . . . . . $q' = .353194$,
for $z = 1.145$ . . . . . . $q' = .356124$.

Finding by interpolation from this the value of z corresponding to $q' = .353553$

we have $$z = -1.1441225,$$

and $$y = z\sqrt{8} = -3.236067,$$

hence $$x = y - 1 = -4.236067.$$

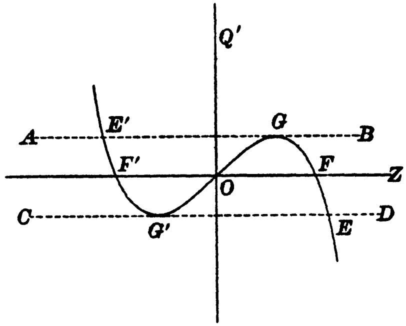
This is the method employed. The *reasons* follow:

The equation being in the form

$$z^3 - z + q' = 0,$$

we may by assigning "consecutive" values to z obtain corresponding values

for the arbitrary constant q', and plotting the result we obtain the curve whose ordinate is q'



Now examine the curve for maximum and minimum points.

From
$$z^3 - z + q' = 0$$
we have by differentiation
$$-\frac{dq'}{dz} = 3z^2 - 1;$$
and for a maximum or minimum point,
$$\frac{dq'}{dz} = 0 = 3z^2 - 1;$$
whence $z = \pm \frac{1}{3}\sqrt{3}$ and the corresponding values q' are $q' = \pm \frac{2}{9}\sqrt{3}$.

Draw the lines AB and CD parallel to the axis of z. Their equations are, respectively
$$q' = +\frac{2}{9}\sqrt{3},$$
$$q' = -\frac{2}{9}\sqrt{3}.$$

Now revert to the condition of applicability of Cardan's Rule
$$\frac{q'^2}{4} - \frac{1}{27} \geqq 0.$$

This gives q' a value greater numerically than or equal numerically to $\frac{2}{9}\sqrt{3}$. Then, in our curve, the division between solubility and insolubility by Cardan's Rule is along the lines AB and CD.

It is interesting to note that for values of q' numerically greater than $\frac{2}{9}\sqrt{3}$, there is only one real value of z, while for values of q' numerically less than $\frac{2}{9}\sqrt{3}$, there are three real values of z (this being all in reference to the curve), and at the same time to remember that Cardan's Rule is applied in practice only to equations having one real root and two imaginary roots, or two equal roots.

Hence we confine ourselves to the portion of the curve between the lines AB and CD, the "Irreducible Case." Observing the form of the curve, we see that by selecting any one of the six arcs EF, FG, GO, $E'F'$, $F'G'$, $G'O$, z may be given values in the selected segment, at small regular intervals, from which corresponding values of q' may be obtained, since $z^3 - z = -q'$. If a sufficient number of such values be obtained a table will result, from which, either z or q' being given, the other may be found by interpolation to any desired degree of accuracy, the accuracy depending upon the magnitude of the interval between successive values of z.

I have selected the arc $E'F'$ because in that portion of the curve q' changes more rapidly than z. This is an advantage, q' being the argument with which the table is generally entered and z the value to be found. The sign of q' does not affect the validity of the table if we remember that the sign of z is always to be taken opposite to that of q', since reversing the signs of z and q' does not affect the equation $z^3 - z + q' = 0$. Hence, whatever the sign of q', provided its value lies between 0 and $\pm\frac{2}{9}\sqrt{3}$, the table above indicated gives one value of z corresponding to the value of q'.

Having found from the table a value of z lying on $E'F'$, taking care to prefix the proper sign, the two other values may be found from a quadratic equation, or similar tables may be constructed for the segments OG and GF.

The table attached includes values of z from 1.000 to 1.160, and gives quite accurate results. It will enable one to solve any cubic equation with real coefficients that is "irreducible."

TABLE FOR THE SOLUTION OF IRREDUCIBLE CASES OF THE CUBIC EQUATION.

z	0	1	2	3	4	5	6	7	8	9
1.00	.000000	.002003	.004012	.006027	.008048	.010075	.012108	.014147	.016193	.018244
1.01	.020301	.022364	.024434	.026509	.028591	.030678	.032772	.034872	.036978	.039090
1.02	.041208	.043332	.045463	.047599	.049742	.051891	.054046	.056207	.058374	.060547
1.03	.062727	.064913	.067105	.069303	.071507	.073718	.075935	.078158	.080387	.082622
1.04	.084864	.087112	.089366	.091627	.093893	.096166	.098445	.100731	.103023	.105321
1.05	.107625	.109936	.112253	.114576	.116905	.119241	.121584	.123932	.126287	.128648
1.06	.131016	.133390	.135770	.138157	.140550	.142950	.145355	.147768	.150186	.152612
1.07	.155043	.157481	.159925	.162370	.164833	.167297	.169767	.172244	.174727	.177216
1.08	.179712	.182214	.184723	.187239	.189761	.192289	.194824	.197366	.199913	.202468
1.09	.205029	.207597	.210171	.212751	.215339	.217932	.220533	.223140	.225753	.228373
1.10	.231000	.233633	.236273	.238920	.241573	.244233	.246899	.249572	.252252	.254938
1.11	.257631	.260331	.263037	.265750	.268470	.271196	.273929	.276669	.279415	.282168
1.12	.284928	287695	.290468	.293248	.296035	.298828	.301628	.304435	.307249	.310070
1.13	.312897	.315731	.318572	.321420	.324274	.327135	.330003	.332878	.335760	.338649
1.14	.341544	.344446	.347355	.350271	.353194	.356124	.359060	.362004	.364954	.367911
1.15	.370875	.373846	.376824	.379809	.382800	.385799	.388804	.391817	.394836	.397863
1.16	.400896									

NOTE.— The first column contains values of z. The numbers 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 at the heads of the other columns belong to z, carrying its value to thousandths. All the columns but the first contain values of q'. The sign of z is always the opposite of that of q'.

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